

*Theorem 4.1 (Laplace Expansion Theorem):* Let  $A$  in  $\mathbb{R}^{n \times n}$ . Then for any  $i$  in  $1, \dots, n$  and  $j$  in  $1, \dots, n$  we have

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij}) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij})$$

where  $\tilde{A}_{ij}$  is the  $(n-1) \times (n-1)$  matrix obtained by deleting row  $i$  and column  $j$  from  $A$ .

The theorem above states that **determinant of a matrix can be computed by a cofactor expansion across any row or down any column.**

*Example 4:* Use Theorem 4.1 to quickly calculate  $\det(A)$  where  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ .

$$|A| = 1(-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 0(-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 0(-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$|A| = -1(-1-1) = -1(-2) = \boxed{2}$$

*Example 5:* Use Theorem 4.1 to quickly calculate  $\det(A)$  where  $A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -5 & 4 & 8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$ .

$$|A| = 2(-1)^{2+3} \begin{vmatrix} 4 & 3 & -5 \\ 7 & 4 & 8 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$$

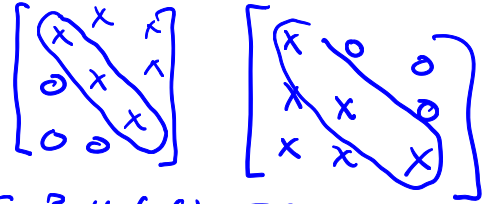
$$= -2 \left( 3(-1)^{2+2} \begin{vmatrix} 4 & -5 \\ 5 & -3 \\ 0 & 2 \end{vmatrix} \right) = -6 \left( 4(-1)^2 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + 3(-1)^{2+1} \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} - 0 \right)$$

$$= -6 \left( 4(4-3) - 5(6-5) \right)$$

$$= -6(4-5) = -6(-1) = \boxed{6}$$

*Definition:* A matrix  $A$  in  $\mathbb{R}^{n \times n}$  is called *upper triangular* if all entries lying below the diagonal entries are zero (i.e.  $a_{ij} = 0$  whenever  $i > j$ ). A matrix  $A$  in  $\mathbb{R}^{n \times n}$  is called *lower triangular* if all entries lying above the diagonal entries are zero (i.e.  $a_{ij} = 0$  whenever  $i < j$ ). A matrix that is both upper triangular and lower triangular is *diagonal*.

*Example 6:* Calculate  $\det(A)$  where  $A = \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & -6 \end{bmatrix}$ .

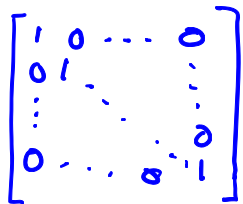


$$|A| = -3(-1)^{1+1} \begin{vmatrix} 4 & 5 \\ 0 & -6 \end{vmatrix} = -3(4(-6) - 0 \cdot 5) = -3 \cdot 4 \cdot (-6) = 72.$$

*Theorem 4.2:* If  $A$  in  $\mathbb{R}^{n \times n}$  is an upper triangular, lower triangular, or diagonal matrix. Then

$$\det(A) = a_{11}a_{22} \cdots a_{nn}$$

*Note:* By theorem 2,  $\det(I_n) = 1$ .



$$\det(I) = 1 \cdot 1 \cdot 1 \cdots 1 = 1$$