> *Theorem 4.1 (Laplace Expansion Theorem):* Let A in $\mathbb{R}^{n\times n}$. Then for any i in $1,\dots,n$ and j in $1,\dots,n$ we have
>
> $$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\tilde{A}_{ij}) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(\tilde{A}_{ij})$$
>
> where $\tilde{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j from A.

The theorem above states that determinant of a matrix can be computed by a cofactor expansion across any row or down any column.

Example 4: Use Theorem 4.1 to quickly calculate $\det(A)$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

$$|A| = 1(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \cancel{0(-1)^{2+2}\begin{vmatrix} \cdots \end{vmatrix}}^{\,0} + \cancel{0(-1)^{3+2}\begin{vmatrix} \cdots \end{vmatrix}}^{\,0}$$

$$|A| = -1(-1-1) = -1(-2) = 2$$

Example 5: Use Theorem 4.1 to quickly calculate $\det(A)$ where $A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & 8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$.

$$|A| = 2(-1)^{2+3}\begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & 8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= -2\left(3(-1)^{2+2}\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}\right) = -6\left(4(-1)^{2}\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + 3(-1)^{2+1}\begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} - 0\right)$$

$$= -6\left(4(4-3) - 5(6-5)\right)$$

$$= -6(4-5) = -6(-1) = 6$$

Definition: A matrix A in $\mathbb{R}^{n\times n}$ is called *upper triangular* if all entries lying below the diagonal entries are zero (i.e. $a_{ij} = 0$ whenever $i > j$). A matrix A in $\mathbb{R}^{n\times n}$ is called *lower triangular* if all entries lying above the diagonal entries are zero (i.e. $a_{ij} = 0$ whenever $i < j$). A matrix that is both upper triangular and lower triangular is *diagonal*.

Example 6: Calculate $\det(A)$ where $A = \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & -6 \end{bmatrix}$.

$$\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \qquad \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

$$|A| = -3(-1)^{1+1}\begin{vmatrix} 4 & 5 \\ 0 & -6 \end{vmatrix} = -3\left(4(-6) - 0\cdot 5\right) = -3\cdot 4\cdot(-6) = 72.$$

Theorem 4.2: If A in $\mathbb{R}^{n\times n}$ is an upper triangular, lower triangular, or diagonal matrix. Then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

Note: By theorem 2, $\det(I_n) = 1$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad \det(I) = 1\cdot 1\cdot 1\cdot 1\cdots 1 = 1$$